DE PRINCIPIIS

A Prolegomenon to Mathematical Psychology

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I.

The empirical sciences, characterized by the possibility of systematic observation and experimentation, are classified under two headings: natural and social. The former supposedly deal with inanimate and nonhuman animate aspects of nature, such as physics, geology, biology, whereas the latter specifically deal with man and all the products of interaction between man and man. Another way of classifying sciences is with reference to the mode of attack: observational and experimental. Objects of the universe which are beyond the capacity of man and his techniques to bring within the bounds of laboratory control can only be "observed"; systematic biology and all the social sciences are instances of this. In particular, social sciences have to presume as a fundamental fact the variability of human nature and will, which challenge the validity of laboratory techniques. Psychology, however, has undertaken a herculean task in the attempt to make an experimental science of itself. Since WUNDT's founding of the psychological laboratory in Leipzig in 1879, innumerable laboratories all over the world have been started and diverse aspects of human nature are being studied in this spirit. After several decades of assiduous experimentation, it is interesting to be told by a notable contemporary psychologist, "In a sense it is true to say that through all this vast mélange the very birth-cry of the infant science is still resounding"[2].

The reason for this inability to advance must be sought in certain methodological errors. Firstly, psychology has all along been overwhelmed by syntax language and has not been able to develop "object" language. The vocabulary is mostly drawn from speculative philosophy, with frequent eccentricities of subjective interpretation. All the principal categories of present-day psychology are at least as old as ARISTOTLE, and the history of the subject is just a succession of restatements of the same issues. Frequently the dictionary fallacy of circular definition is made.

Secondly, DANTZIG's criticism[3] that philosophy lacks the "principle of relativity" holds true for psychology also. The scope of psychology has never been properly defined and the absence of a particular frame of reference has deprived the experimental results of integrity. The confusion that prevails in this subject, although often unrecognized, is evidenced by the multitude of enunciations of the subject matter that exist.

Thirdly, the zeal for experimentation has resulted in the mass production of assertions, of which no "propositional function" exists; the particular constituents never necessitate generalizations, the instances seldom lead to concepts and the subject lacks "theoretical certainty"; the experimental findings do not bear the stamp of finalty, and do not admit of universal application.

Finally, as BRIDGMAN notes, "In the social sciences there is lacking, to such a large extent as to make a difference in the general atmosphere, that disinterested point of view which in the physical sciences we associate with so-called pure science as distinguished from applied science"[1]. Practical application is the temptation under which most psychological research is conditioned or hurried through.

In effect, the subject of psychology lacks the logical structure which is essential to all science. One of the essential tasks of scientific endeavour is to determine logical constants which will help systematization and classification of categories and an exact analysis of facts. The spirit of logic or mathematics must be infused into psychology if it is to preserve its integrity.

II.

Experience convinces us that objects and events, and sets of objects and events, often obey logical schemes. This assumption is at the back of all science: experimentation and measurement. The method of representing logical schemes and working out their implications characterizes mathematics; it is a pure hypothetico-deductive system. "We understand the term *mathematical science to mean any set of propositions arranged according to a sequence of logical deduction*"[2]. There must, in the first instance, be a set of axioms or postulates consisting of one or more undefined primitives, and secondly there must be rules of deduction or a system of logic. The axioms need not be obvious truths, as BERTRAND RUSSELL points out, but the set of axioms must be consistent, and each axiom must be independent and categorical. As NEWSON notes, "axiom written in the acceptable form has the form of a proposition and the characteristics of a mathematical function." It has the function of a "theoretical juice extractor" as HEMPEL describes it. The actual extractions are done by the formal deductive reasoning, or logical analysis.

What is the province of mathematics? POINCARÉ wrote: "Mathematics do not study objects but the relation between objects. Matter does not engage their attention. They are interested in form alone." Nothing new, therefore, is discovered or added to the content of our knowledge. It is just a conceptual technique for the *scientific understanding* of our experiential data. Mathematics is essentially a language: to represent the structures of objects, structures of symbols are built and the "structural properties of symbolic systems" are studied. Thus mathematics helps crystallization of thought and necessitates the fruitful analysis.

III. *Groundwork*

(1) An attempt is made here to develop psychology on the basis of a small number of sufficient and necessary assumptions or axioms. Before enunciating some axioms, a passing reference must be made to some of the most fruitfully applicable mathematical concepts and conventions.

A set is a collection of elements or individuals, wherein a definition obtains determining the "belongingness" or membership character of any particular individual or element with respect to the set. When A and B are

[1] Indian Institute of Science, Bangalore, India.

[2] G. MURPHY, *An Historical Introduction to Modern Psychology* (Routledge & Paul Kegan, London, 1949), p. 17.

[3] T. DANTZIG, *Number, the Language of Science* (Allen & Unwin, London, 1942), p. 232.

[1] P. W. BRIDGMAN, *Reflections of a Physicist* (Philosophical Library, New York, 1950), p. 304.

[2] D. VEBLEN and J. W. YOUNG, *Projective Geometry*, Vol. I (Ginn & Co., Boston, 1910), p. 2.

two sets, and x any element, we have the following possible relations:

$A = B$, i.e., $A \subset B$ and $B \subset A$ (identity);
$A \subset B$, or $B \supset A$ (inclusion);
$x \in A$ (membership);
$A \not\subset B$, $A \not\supset B$, $A \neq B$, $x \notin A$ (negation);
$A \cap B$ (product);
$A \cup B$ (union);
$A - B$ (remainder);
$A = 0$ (null set);
$x \in (a)$, i.e., $x = a$ (one-element set);
$A \cap B = 0$ (empty intersection).

All the above concepts have been used in their usual topological sense. Among the axioms that obtain, we note:

If x is an element and S any set:

(*a*) When $x \in S$, U is a set of elements containing x (HAUSDORFF). Call U the neighbourhood of x.

(*b*) If $x \in U$ and if y is also an element of U, then U is the neighbourhood of y also.

(*c*) If U and V are neighbourhoods such that $x \in U \cap V$ then there exists a neighbourhood W such that $W \subset U \cap V$.

(*d*) If x and y are distinct elements, then there exists a neighbourhood of x not containing y.

The set of elements that proves true to these axioms is termed a topological space. The set theoretical method studies the local properties of this space, such as positional relationships, connectivity, belongingness and qualitative definition. The locally connected continuum is defined by the Jordan curve, which decomposes the plane into precisely two separated regions, one lying inside and the other outside the curve.

The other topological concepts that are employed in this customary significance are such as closed or open sets, bounded sets, complementary sets, derived sets, limit points, interior or exterior points, boundary points, region, cut, connection, ordering.

(2) We thus arrive at the concept of a space or field—"a totality of possibilities of relative positions of practically rigid bodies" (EINSTEIN). But in order to overcome the limitations of treating psychological phenomena as rigid or static, vector view-point is necessary. The psychological space is a dynamic whole comprehending both function and structure, entity and event. The psychological space is a vector field, for the dynamic property of this space originates and determines psychological action: the field forces are the vectors. A vector produces and directs psychological action. It implies a movement from an origin (A) to a terminus or goal (B), thus $\overrightarrow{AB} = \alpha$. If the movement is negatively valenced, we get $\overleftarrow{AB} = -\alpha$. Certain assumptions obtain herein:

(*a*) Every pair of ordered points A, B whether coincident or not, determines a vector $\overrightarrow{AB}$; and if A is any point and α any vector, there exists only one point B such that $\overrightarrow{AB} = \alpha$.

(*b*) Each pair of vectors α, β determines uniquely a vector γ such that if $\overrightarrow{AB} = \alpha$, and $\overrightarrow{BC} = \beta$, then $\overrightarrow{AC} = \gamma$.

(*c*) Of two vectors α, β, other things being equal, the vector of greater magnitude tends to suppress the vector of lesser magnitude.

(*d*) Of two vectors α, β, other things being equal, the one that easily tends to greatest possible satisfaction, tends to suppress the other that does not.

(3) We can also fruitfully borrow, *mutatis mutandis*, the concept of FRÉCHET's metric space, where the distance function $\delta(x, y)$ satisfies the following conditions

(*a*) $\delta(x, x) = 0$.
(*b*) If $x \neq y$, then $\delta(x, y) > 0$;
(*c*) $\delta(x, y) = \delta(y, x)$;
(*d*) $\delta(x, y) + \delta(y, z) > \delta(x, z)$.

Thus by the utilization of the concept of force (vector) and distance, it is possible to evolve the theory of psychological work. KURT LEWIN has indeed shown the way in which topological and vector concepts might be used for psychological research.

IV. *Psychological Axiomatics*

(1) We will state a set of three axioms, of inclusion, attribution and action respectively.

(i) If S is any point-set, and m any member thereof, m and S are related by inclusion such that

(*a*) $m = m$; or $m \in (m)$,
(*b*) $m \neq S$; or $S - m \neq 0$;
(*c*) $\bar{m} \in S$.

(ii) Whenever $m \in S$, the relation of inclusion satisfies the following conditions:

(*a*) m has the attribute of m; i.e., $m \overset{\hat{m}}{\simeq}$,

(*b*) m has no attribute of $\bar{m}$; i.e., $m \overset{\hat{\bar{m}}}{\not\simeq}$,

(*c*) m has the attribute of S; i.e., $m \overset{\hat{S}}{\simeq}$.

(iii) Whenever $m \in S$, it is always an action-system such that

(*a*) m acts in accordance with $m \overset{\hat{m}}{\simeq}$; i.e., $m \overset{\hat{m}}{\rightarrow}$,

(*b*) m acts not in accordance with $m \overset{\hat{\bar{m}}}{\simeq}$; i.e., $m \overset{\hat{\bar{m}}}{\nrightarrow}$,

or/and

(*c*) m acts in accordance with $m \overset{\hat{S}}{\simeq}$; i.e., $m \overset{\hat{S}}{\rightarrow}$.

(1. 1) We will establish the convention of regarding the human being (the locus of our study) as a member of the set of living beings. Human being, we will denote as p, living being as l and the set of human beings as L. By the first axiom, we obtain the proposition:

$$p \mid l \in L$$

which reads "The human being as a living being is a member of the set of living beings". The implied assumption is that there is in L, l other than p ($\bar{p}$), or in other words, p is a distinguished l— although we shall not for the present define the distinction. It is therefore reasonable to assume that there exists a set P such that $p \in L$ for all and only p. It is easy to prove $P \subset L$.

(1. 11) If p_1 is any member of the subset P, there exists a neighbourhood $\mathfrak{f}_1$, a collection of members one of which is p_1. In case $\mathfrak{f}_1$ contains another member $p_2 \in P$, such that p_1, $p_2 \in \mathfrak{f}_1$, we will assume $p_1 \cap p_2 \in \mathfrak{f}_1$. We will call this $\mathfrak{f}_1$, then, the "immediate personal ring" of p_1. If, moreover, $\mathfrak{f}_1 \cap \mathfrak{f}_2 \neq 0$ (where $p_2 \notin \mathfrak{f}_1$ but $p_2 \in \mathfrak{f}_2$), meaning that the intersection of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ is nonempty, or in other words that there is a part of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ which is common to both $\mathfrak{f}_1$ and $\mathfrak{f}_2$, i.e.

$$(p_1 \in \mathfrak{f}_1) \cap (p_2 \in \mathfrak{f}_2) \neq 0,$$

there exists a set $\mathfrak{f}'$ such that $\mathfrak{f}' = \mathfrak{f}_1 \cap \mathfrak{f}_2$. The two neighbourhoods $\mathfrak{f}_1$ and $\mathfrak{f}_2$ of p_1 and p_2 respectively are not mutually exclusive, on assumption, and they overlap to some extent. Again it stands to reason that there is a family of all such $\mathfrak{f}'$ of $p_1(\mathfrak{f}_1)$, $p_2(\mathfrak{f}_2)$, $\ldots$, $p_n(\mathfrak{f}_n)$, namely F. We can convince ourselves that no $p \in F$ is an isolated individual, and further that the set F is connected.

We can continue this process and show that where $F_1 \cap F_2 \neq 0$ there exists F' such that $\mathfrak{f}_\alpha$, $\mathfrak{f}_\beta \subset F'$; and the collection of all F' we will denote $\mathfrak{S}$—which is what we ordinarily term "society" in the sense of community. And, likewise, $\mathfrak{S}' \supset F'_a, F'_b$, where $\mathfrak{S}_1 \cap \mathfrak{S}_2 \neq 0$; and the set of all $\mathfrak{S}'$ we will denote S (or what we ordinarily term "Society"). Thus we can define any p that satisfies these conditions by the following proposition

$$p = l \mid p \in \mathfrak{f}' \subset F' \subset \mathfrak{S}' \subset S \subset L.$$

Remark:—$\mathfrak{S}$ (society) is a relatively concrete entity and is more immediate than S (Society), which is relatively abstract. Note in this context the distinction of TÖNNIES between *Gemeinschaft* and *Gesellschaft*[1]. While it is true in every sense to say $p \in \mathfrak{S}$ it is true to say $p \in S$ only in the sense of primary inclusion; one can see that $p \in F$ is a still more immediate relation than $p \in \mathfrak{S}$. It is conceivable that in this chain of inclusions L is the greatest member, i.e. including every other member, and p is the least member, being included in every other member. It may not be altogether improper to speak of them as the least upper bound (l.u.b.) and the greatest lower bound (gr.l.b.) respectively, and to express the system in a Hasse diagram. It would indeed be interesting to develop a lattice theory in this frame of reference.

(1. 12) We have proceeded above on the assumption of inclusion. We now proceed on the assumption of exclusion. Suppose p_1 and p_2 are two entirely isolated individuals such that $p_1 \cap p_2 = 0$; that is to say, the neighbourhood $\mathfrak{f}_1$ of p_1 does not contain any member of $\mathfrak{f}_2$ of p_2, i.e., $\mathfrak{f}_1 \cap \mathfrak{f}_2 = 0$. It is then obvious that p_2 is an exterior element with respect to p_1. But this assumption will not jeopardize our basic assumption p_1, $p_2 \in L$, nor, p_1, $p_2 \in P$, nor $p \in S$, perhaps not even p_1, $p_2 \in \mathfrak{S}$. Thus it is inevitable that at some stage in the chain, the difference must be substituted by identity, owing to the progressive generality of the properties of the including sets. We may get, e.g., at one stage,

$$p_1 \in \mathfrak{f}_1 \subset F_1 \subset \mathfrak{S} \subset S \subset L$$

and

$$p_2 \in \mathfrak{f}_2 \subset F_2 \subset \mathfrak{S} \subset S \subset L.$$

(1. 2) An inclusion, by the second axiom, implies an attribution. An illustration might clarify the issue. Suppose S is a set consisting of all one-eyed negroes, each of whom, say, is an m. Take any m, say N; he has the characteristics peculiar to himself and to none besides himself, and then he has also the characteristics that are commonly possessed by all m's in S, e.g., being one-eyed. If, however, we consider the class of one-eyed negroes, viz. S, as a subset of the class of negroes, say $\boldsymbol{S}$, the same conditions will apply to this new inclusive class, without disturbing in any way the attributes of N; he will not, for instance, cease to be one-eyed when he is considered as $\in \boldsymbol{S}$, even though $\boldsymbol{S}$ would admit all two-eyed negroes in coexistence with all the one-eyed ones. But he has become more *defined*, in that the attributes he has in common with the two-eyed negroes will now be ascribed to him. Thus, inbetween m, the gr.l.b. (?) and $\boldsymbol{S}$, the l.u.b. (?), any introduction of a new relation of inclusion (and dually of exclusion) serves to determine an attribute, i.e., the individual comes to be characterized by the possession of the attribute of the new set of which that member is a member. Consider, as a further illustration, this series of propositions:

(1) X is a living being (attribute, say, Life).
(2) X is a human being (Humanity).
(3) X is a man (Sex).
(4) X is an Indian (Nationality).
(5) X is a Hindu (Religion).
(6) X is a teacher (Profession).
(7) X is a University teacher (Status).
(8) X is fair-complexioned and well-built (Physical Features).
(9) X is 5 feet 9 inches high (Height).
(10) X is a son of Y (Ancestry).
(11) X is X (Identity).

We start from the absolute generality of the first proposition, which makes no distinction whatever between X and Z, for instance, when X is a man and Z a zebra. But the second proposition is more limited in scope; it distinguishes X from any living being that is not human, thus excluding many members of the original set. As this chain of progressive inclusions proceeds, X is more and more precisely defined, that is, his attributes are one by one disclosed, until the last proposition where the identity of X is entirely fixed (hypothetically). $X_{(11)}$ is included in every one of the preceding sets, and *ipso facto*, is possessed of the attributes of each of them. We may express the chain as:

$$X_{(11)} \subset X_{(10)} \subset \cdots \subset X_{(1)}.$$

It is important, in this connection, to realize the full import of the suggestion of GEORGE BOOLE to identify the attribute and the possessor of the attribute.

(1. 3) An attribution, by the third axiom, involves an appropriate action. We have, in other words, in keeping with a relation of inclusion and the possession of an attribute, a particular mode of action, determined by these two conditions. Each inclusion thus defines an attribute which determines an action. As in attribution, then, even here there is a graded generality of action pattern. Consider, as an illustration, p_1 and p_2 as two different individuals of a set F such that p_1, $p_2 \in F$; we now have

$$\left(p \xrightarrow{\hat{p}_1}\right) \cap \left(p_2 \xrightarrow{\hat{p}_2}\right) = 0,$$

but

$$\left(p_1 \xrightarrow{\hat{p}_1 \& \hat{F}}\right) \cap \left(p_2 \xrightarrow{\hat{p}_2 \& \hat{F}}\right) \neq 0,$$

$\overset{\hat{F}}{\to}$ being in common. And if $p_1 \in F_1$ and $p_2 \in F_2$ but F_1, $F_2 \in \mathfrak{S}$, we have

$$\left(p_1 \xrightarrow{\hat{p}_1 \& \hat{F}_1}\right) \cap \left(p_2 \xrightarrow{\hat{p}_2 \& \hat{F}_2}\right) = 0,$$

but

$$\left(p_1 \xrightarrow{\hat{p}_1 \& \hat{F}_1 \& \hat{\mathfrak{S}}}\right) \cap \left(p_2 \xrightarrow{\hat{p}_2 \& \hat{F}_2 \& \hat{\mathfrak{S}}}\right) \neq 0,$$

$\overset{\hat{\mathfrak{S}}}{\to}$ being in common.

And so on till we get $\overset{\hat{L}}{\to}$ as the common action, i.e., action common to all individuals, whatever their intermediate inclusions. We may thus distinguish between different levels of common action in accordance with inclusion relations and attribute characterizations; $m \overset{\hat{m}}{\to}$ is the least common and $m \overset{\hat{L}}{\to}$ the most common. We might describe the former as individual or peculiar behaviour and the latter as universal behaviour (ordinarily termed "instinctive behaviour"). There is in evidence

[1] F. TÖNNIES, *Fundamental Concepts of Sociology*, translated by C. LOOMIS (American Book Co., New York, 1940).

what is loosely termed "the hierarchy of response mechanism" rendering the process of evolution connected.

(1. 31) We will now assume a lemma: There is for every $m \mid l$ a form-quality (*Gestaltqualität*). By this we mean that there is an action-pattern (or an inherent tendency to act) for every individual member, which is solely conditioned by the attribute system, which in turn is determined by the inclusion relation. We thus assume for each living being, in particular each human being, a dynamic "structure" with properties of its own; we accept also that a p, in particular, is unthinkable without its phenomenal frame of reference. The individual always exists in *Gestalt* and acts in *Gestalt*.

(1. 32) And another lemma: The form-quality operates so that equilibrium exists. We shall not, however, define equilibrium although what it means is obvious enough. Whenever an individual suffers a new inclusion relation and thereby obtains an attribute appropriate to the inclusion, it acts so that this new property may not contradict the previous and original ones and further that it might be integrated in the latter—thus making a structure. Whatever, however, the inclusions, attributes and actions may be, the individual is always recognizable as itself, satisfying axiomatically $p = p$. This is, by assumption, an invariant.

Consider $\mathfrak{P}$ as the *domain* of p; the domain is by definition the saturation of all the inclusions and attributions. Characterize as B the operation of the form-quality in a state of disequilibrium. If, further, $\mathfrak{P}$ is a co-ordinate system, and B the transformation that results in $\mathfrak{P}'$, a new co-ordinate system, we thus get, *mutatis mutandis*, the Wundheilerian equation

$$\mathfrak{P}' = \mathfrak{f}(\mathfrak{P}, B).$$

$\mathfrak{P}'$ is therefore the realization of the form-quality of p, and is describable as the "psychologic object". It is essentially a process of becoming, an emergence, a synthetic event: it is a particular form of equilibrium or a structure (or rather restructurings). The distinction between structure and function is fictitious in the ultimate analysis, although it might serve a useful purpose in the preliminary analyses.

(1. 4) $p \mid l \in F \subset \mathfrak{S} \subset S \subset L$; therefore

$$F - p \neq 0; \quad \mathfrak{S} - F \neq 0; \quad S - \mathfrak{S} \neq 0 \quad \text{and} \quad L - S \neq 0.$$

Thus we get the following chain of complementary relations,

$$F = p \cup \neg p \text{ (read } \neg \text{ as "the complement of")};$$
$$\mathfrak{S} = p \cup \neg p \cup \neg (p \cup \neg p);$$
$$S = p \cup \neg p \cup \neg \{\neg (p \cup \neg p)\};$$

and

$$L = p \cup \neg p \cup \neg [\neg \{\neg (p \cup \neg p)\}];$$

or more simply

$$L = p \cup \neg p \cup \neg F \cup \neg \mathfrak{S} \cup \neg S.$$

That is to say,

$$L - p = \neg p \cup \neg F \cup \neg \mathfrak{S} \cup \neg S$$

or $L - p = E$ (where E symbolizes Environment). It is obvious then, that $E = \neg p$, for

$$p \in L, \quad E \subset L, \quad \text{and} \quad p \notin E \quad \text{but} \quad L = p \cup E.$$

It follows also, $p = \neg E$.

(1. 41) Translate p into $\mathfrak{P}$, and E into $\mathfrak{E}$ where $\mathfrak{P}$ and $\mathfrak{E}$ are the saturations of all the inclusions and attributions of p and E respectively, such that $\mathfrak{P} \subset \mathfrak{E}$. We thus have

$$\mathfrak{P} = p \cap F \cap \mathfrak{S} \cap S \cap L,$$

and

$$\mathfrak{E} = (F \cap \mathfrak{S} \cap S \cap L) \cup (\mathfrak{S} \cap S \cup L) \cup (S \cap L).$$

(1. 42) $\mathfrak{P}$ is, by definition, not a class and is therefore a unit; but it is a complex unit or a whole, "the parts of which are other units which are presupposed in it". It is, in other words, structured. As $\mathfrak{E}$ is also a structured set, it follows from the above axioms:

(*a*) the structure of $\mathfrak{P}$ is unique in itself;

(*b*) the structure of $\mathfrak{P}$ is consequent on the structure of $\mathfrak{E}$;

(*c*) the action in the structure of $\mathfrak{P}$ is contingent on the structure of $\mathfrak{E}$.

Prospect

The above study is an inquiry into the possibility of constructing an independent mathematical discipline for psychology, having its own postulational technique and deductive logic. Preliminarily and tentatively three axioms (of inclusion, attribution and action) have been enunciated and their implications developed. It is suggested that further postulates are necessary and a body of lemmas, theorems and propositions can be hoped to be evolved. Needless to say this is a very humble beginning, and being new, suffers from many limitations. But shall we not remember the query of KEPLER: "What is the use of a new-born babe?"

Zusammenfassung

Der Gegenstand der Psychologie ist mit zahlreichen ihm innewohnenden Fehlerquellen sowohl logischer wie auch methodischer Art durchsetzt, was wirklichen Fortschritt auf diesem Gebiet, sowie die Erlangung von wissenschaftlichem Status, verhindert hat. Um diesem Mangel abzuhelfen, ist der Versuch unternommen worden, eine unabhängige mathematische Disziplin zu entwickeln, die imstande wäre, psychische Erscheinungen zu beschreiben und zu erklären. Drei Axiome: das der Einschliessung, das der Zuschreibung und das der Handlung sind vorläufig aufgestellt worden. Es ist als erstes Ergebnis dieser Bemühung möglich geworden, den «psychischen Gegenstand» festzustellen und zu beschreiben, was ja die unentbehrliche Voraussetzung dieses Erkenntniszweiges ist. Es ist zu vermuten, dass die oben angedeuteten Postulate und Übereinkünfte zu weiteren Haupt-, Lehr- und Hilfssätzen führen werden, die dem Aufbau einer wissenschaftlichen Psychologie dienen können.